

Beauty of Mathematics Decal PSET #2 Solutions

Due 09/27

Recall that an infinite quantity is said to be “countable” if it is the same size as the whole numbers $\{0, 1, 2, 3, \dots\}$ and “uncountable” if it is bigger (i.e. not countable).

1. Show that an uncountable set can't be made up of two countable sets. For example, if we have a countable subset of the real numbers, then the rest of the real numbers must be uncountable, since the whole set of real numbers is uncountable. (Hint: the even numbers and the odd numbers are countable, and they make up the whole numbers, which are also countable).

Solution: If we have two countable sets, then we can match one of them up with the even whole numbers $\{0, 2, 4, 6, \dots\}$ and match the other up with the odd whole numbers $\{1, 3, 5, \dots\}$. Then the two sets together are matched up with the whole numbers $\{0, 1, 2, 3, \dots\}$, which shows that the two sets together are countable.

2. Suppose we have an “alphabet” consisting of finitely many symbols. How many “sentences” are there, if a “sentence” is a finite string of symbols from our alphabet? (Possible hint: alphabetical order).

Solution: We can put an order on our alphabet, and use this to define an order on the set of sentences. Namely, shorter sentences come first, and we order sentences of the same length by alphabetical order. Then we can use this order to make a matching between the set of sentences and the whole numbers: match the first sentence with 0, the second sentence with 1, the third with 2, etc. This shows that there are countably infinitely many sentences.

3. Call a real number *computable* if it's possible to write down a (finite) program which, when run, will output the digits of this number. For a trivial example, the number 1 is computable, because our program can just be `output 1`. For a less trivial example, the number $0.101001000100001\dots$ is computable, because we could write something like `for each whole number $n \geq 1$, output a 1 followed by n zeros`. How many real numbers are computable? How many real numbers are not computable? (Hint: use the previous exercise for the first question and the previous previous exercise for the second question).

Solution: Computer programs are written from a finite list of symbols, and a program is a finite string of such symbols, so the previous exercise shows that there are countably many computer programs. The number of computable numbers can be no bigger than the number of programs, so there are countably many computable real numbers.

By the first exercise we can conclude that there are uncountably many real numbers that are not computable. That is, almost all real numbers can't be written down—even in theory!

(Note that being computable is not the same as being rational; in the second example above, $0.101001000100001\dots$ is irrational.)